

Anomalous Dimensions of Baryon Multiplets in $SU(N)$ ($N \geq 3$) Flavor Symmetry

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Abstract

The QCD evolution equation for the antisymmetric flavor multiplet is solved in $SU(N)$ ($N \geq 3$) flavor symmetry. This work presents the leading anomalous dimensions of all possible baryon multiplets in $SU(N)$ flavor symmetry. We observe that the quark distribution amplitudes of all baryon states are expanded in terms of Appel polynomials, $A_i(x)$, and only the antisymmetric polynomials (e.g. A_1 and A_4 when $i \leq 5$) have non-zero coefficients for the antisymmetric flavor $\left\{ \frac{N(N-1)(N-2)}{6} \right\}$ multiplet. For $N=3$, this observation provides a constraint for building the model quark distribution of Λ_1 baryon. The asymptotic high Q^2 behavior of the Λ_1 form factor is also presented.

1 Introduction

The form of the short-distance behavior of a baryon wavefunction can be computed systematically in perturbative quantum chromodynamics (QCD) [1]. The leading behavior of the baryon three-quark wave function at large momentum transfer is controlled through an evolution equation with an irreducible hard scattering kernel which, in lowest order, is identical to the gluon-exchange potential. Since the running coupling constant $\alpha_s(Q^2) = 4\pi / [\beta \ln(Q^2/\Lambda^2)]$ ($\beta = 11 - \frac{2}{3}n_f$, where n_f is the number of flavors) is small for large momentum transfer Q , a perturbative calculation of the short-distance part of the wave function can be justified. By solving the evolution equation, one can find the anomalous dimensions which control the short distance behavior of the wave functions. The obtained anomalous dimensions can be used in the analysis of the exclusive processes involving the large momentum transfers. For example, the detailed Q^2 -dependence of the baryon form factor at the asymptotic large Q^2 region can be predicted by determining the anomalous dimensions of the three-quark amplitude. It has also been shown that the anomalous dimensions of the three-quark amplitude can be predicted by the operator-product expansion and the renormalization group [2].

A particularly convenient and physical formalism for studying processes with large momentum transfer is light-cone quantization [1]. A systematic analysis of exclusive processes and hadron distribution amplitudes has been given, including solutions of the evolution equations of the three-quark system. Recently, a general method for solving the QCD evolution equations which govern relativistic multiquark wave functions was presented [3]. In the case of three-quark systems, a light-cone basis of completely antisymmetric wave functions was generated and a distinctive classification of nucleon and delta wave functions was obtained. The corresponding Q^2 dependence of the baryon distribution amplitudes distinguishes the nucleon and delta form factors. Since the QCD evolution kernel can commute with the raising and lowering operators of the $SU(N)$ flavor group, one can easily show that the anomalous dimensions obtained for the nucleon and delta are actually those for the $\left\{ \frac{N(N^2-1)}{3} \right\}$ multiplet and $\left\{ \frac{N(N+1)(N+2)}{6} \right\}$ multiplet in $SU(N)$ flavor symmetry. Thus far, since the analysis of evolution equations was limited to $N=2$, the totally antisymmetric flavor multiplet which appears in $N \geq 3$ has not yet been studied. In particular, the anomalous dimensions of the baryon singlet, Λ_1 , ($N=3$) are not yet been obtained.

In this paper, we apply the general method developed in previous literatures [3, 4] to the baryon singlet and present the anomalous dimensions of the totally antisymmetric flavor multiplet (i.e. $\left\{ \frac{N(N-1)(N-2)}{6} \right\}$ multiplet in $SU(N)$). This work presents the leading anomalous dimensions of all possible multiplets in $SU(N)$ flavor symmetry which is an exact symmetry in the limit of large momentum transfer or the short distance limit of three-quark wave functions.

In Section 2, we construct completely antisymmetric representations of the baryon singlet. In Section 3, we operate the QCD evolution kernel to the constructed antisymmetric basis and present anomalous dimensions of $\left\{ \frac{N(N-1)(N-2)}{6} \right\}$ multiplet in $SU(N)$ flavor symmetry. Discussions and future directions are followed in Section 4.

2 Baryon Singlet Representations

As described in the general method [3], we first construct completely antisymmetric representations of the baryon singlet. A fermionic system in QCD is classified by the assignment of four quantum numbers: color (C), flavor (T), spin (S), and orbital (O). Each quantum sector of the wave function can be classified using irreducible representations with permutation symmetry denoted by Young diagrams. Since the baryon singlet has the totally antisymmetric representation both for color and flavor;

$$\begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}_C = \frac{1}{\sqrt{6}}(ryb + ybr + bry - byr - rby - yrb), \quad (2.1a)$$

$$\begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}_T = \frac{1}{\sqrt{6}}(uds + dsu + sud - sdu - usd - dus), \quad (2.1b)$$

the spin-orbit representation must also be totally antisymmetric;

$$\begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}_{SO} = \begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}_S \times \begin{array}{c} \boxed{} \\ \boxed{} \end{array}_O .$$
(2.2)

This construction is consistent with the fact that the SU(3) flavor singlet Λ is forbidden by Fermi statistics in the ground state multiplet. Therefore, the flavor singlet Λ_1 discussed in this paper should have the orbital angular momentum larger than zero (i.e. $L \geq 1$). The orbital states are normally defined by the quantum numbers of angular momentum L and L_z . On the light cone, the quark distribution amplitude $\phi(x_i, Q)$ is defined by

$$\begin{aligned}
 \phi(x_i, Q) &= \left[\ln \frac{Q^2}{\Lambda^2} \right]^{-3C_F/2\beta} \\
 &\times \int^Q \left[\prod_{i=1}^3 \frac{d^2 \vec{k}_{\perp i}}{16\pi^3} \right] 16\pi^3 \delta^2 \left[\sum_{i=1}^3 \vec{k}_{\perp i} \right] \psi^{(Q)}(x_i, \vec{k}_{\perp i}),
 \end{aligned}$$
(2.3)

where $\psi^{(Q)}(x_i, \vec{k}_{\perp i})$ is the wave function of three quark which has longitudinal momentum fractions $x_i = k_i^+ / (\sum_{i=1}^3 k_i^+) = (k_i^0 + k_i^3) / [\sum_{i=1}^3 (k_i^0 + k_i^3)]$ and transverse momenta $\vec{k}_{\perp i}$. The term $3C_F/2\beta$ ($C_F = (n_C^2 - 1)/2n_C = 4/3$) in Eq.(2.3) is due to the wave-function renormalization of the quark propagators. In this definition, the $L_z=0$ projection defines the amplitude for finding the constituents collinear up to the scale Q . We will use as a basis for the orbital dependence of $\phi(x_i, Q)$ the index-power space representations [3] $x_1^{n_1} x_2^{n_2} x_3^{n_3}$ with $n = n_1 + n_2 + n_3$. The total power is analogous, as far as permutation symmetry is concerned, to the angular momentum L for the nonrelativistic system. In the QCD evolution equation the minimal anomalous dimensions γ_n which determine hadronic amplitudes at very short distances are associated with small values of n ; only the smallest powers of x_i are important for probing the short-distance behavior of $\phi(x_i, Q)$. Thus, we consider the "orbital" symmetry on the index-power space ($n = n_1 + n_2 + n_3$) which determines the power of x_1, x_2 and x_3 such as $x_1^{n_1} x_2^{n_2} x_3^{n_3}$.

In this power space, the orbital states are determined by filling up the possible Young diagrams with the powers of x_i . As in the previous literature [3, 4], we will consider the explicit representations and Young diagrams up to $n=2$. By considering the Clebsch-Gordon coefficients of S_3 permutation group, we obtain the following spin-orbit representations;

$$\begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}_{SO} = \begin{cases} \sqrt{\frac{21 \times 5!}{6}} \{ \downarrow \uparrow \uparrow (-x_2 + x_3) + \uparrow \downarrow \uparrow (x_1 - x_3) \\ + \uparrow \uparrow \downarrow (-x_1 + x_2) \}, & \text{for } n = 1 \quad (2.4a) \\ \frac{6\sqrt{28} \times \sqrt{5!}}{\sqrt{6}} [\downarrow \uparrow \uparrow \{(-x_1 x_3 + x_1 x_2) + \frac{1}{4}(-x_2 + x_3)\} \\ + \uparrow \downarrow \uparrow \{(x_2 x_3 - x_1 x_2) + \frac{1}{4}(x_1 - x_3)\} \\ + \uparrow \uparrow \downarrow \{(-x_2 x_3 + x_1 x_3) + \frac{1}{4}(-x_1 + x_2)\}] , & \text{for } n = 2 \quad (2.4b) \end{cases}$$

From this, we can easily see that the baryon singlet has only two Appel polynomials $A_1(x)$ and $A_4(x)$ when $n \leq 2$. Multiplying the explicit representations given by Eqs. (2.1) and (2.4), we obtain the totally antisymmetric baryon singlet representations;

$$\begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}_{CTSO} = \begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}_C \times \begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}_T \times \begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \end{array}_{SO} \quad (2.5)$$

In terms of Appel polynomials, the baryon singlet, octet and decouplet representations of SU(3) flavor up to $n=2$ are summarized by Table I. The representations of nucleon and delta in Table I are the same as those given in Ref [3].

3 The Evolution Equation and Solution

The evolution equation for the three-quark distribution amplitude $\phi(x_i, Q)$ with $L_z=0$ is given by

$$x_1 x_2 x_3 \left[\frac{\partial}{\partial \xi} + \frac{3C_F}{2\beta} \right] \bar{\phi}(x, Q) = \frac{C_B}{\beta} \int_0^1 [dy] V(x, y) \bar{\phi}(y, Q), \quad (3.1)$$

where the reduced amplitude $\tilde{\phi}(x, Q)$ and the variable ξ are defined by

$$\phi(x, Q) = x_1 x_2 x_3 \tilde{\phi}(x, Q), \quad (3.2)$$

and

$$\xi(Q^2) = \frac{\beta}{4\pi} \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \alpha_s(k^2) \sim \left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)} \right] \quad (3.3)$$

The color factor $C_B = (n_c + 1)/2n_c = \frac{2}{3}$ is fixed. The evolution kernel $V(x, y)$ is the sum over interactions between quark pairs i, j due to exchange of a single gluon:

$$V(x, y) = x_1 x_2 x_3 \tilde{V}(x, y) \quad (3.4)$$

and $\tilde{V}(x, y) = \tilde{V}_\delta(x, y) + \tilde{V}_\Delta(x, y)$ is given by

$$\tilde{V}_\delta(x, y) \equiv \int_0^1 [dy] \sum_{i \neq j} \theta(y_i - x_i) \delta(x_k - y_k) \frac{y_j}{x_j} \frac{\delta_{h_i h_j}}{x_i + x_j}, \quad (3.5a)$$

$$\tilde{V}_\Delta(x, y) \equiv \int_0^1 [dy] \sum_{i \neq j} \theta(y_i - x_i) \delta(x_k - y_k) \frac{y_j}{x_j} \frac{\Delta}{y_i + x_i}, \quad (3.5b)$$

where $\delta_{h_i h_j} = 1(0)$ when the helicities of quark pairs i, j are antiparallel (parallel). The infrared singularity at $x_i = y_i$ is cancelled by $\Delta\tilde{\phi}(y, Q) \equiv \tilde{\phi}(y, Q) - \tilde{\phi}(x, Q)$ reflecting the fact that the baryon is a color singlet. The evolution equation Eq. (3.1) has a general solution of the form

$$\phi(x, Q) = x_1 x_2 x_3 \sum_{n=0}^{\infty} C_n \tilde{\phi}_n(x) \left[\ln \frac{Q^2}{\Lambda^2} \right]^{-\gamma_n}, \quad (3.6)$$

where γ_n and $\tilde{\phi}_n$ satisfy

$$\left(\frac{3C_F}{2\beta} - \gamma_n \right) \tilde{\phi}_n(x) = \frac{2C_B}{\beta} \int_0^1 [dy] \bar{V}(x, y) \tilde{\phi}_n(y). \quad (3.7)$$

Setting the y representations of $\tilde{\phi}_1$ and $\tilde{\phi}_4$ [See Eqs. (2.4) and (2.5), and Table I(E)] into the three-quark evolution equation [Eqs. (3.4), (3.5) and (3.7)], we find

$$\int_0^1 [dy] \bar{V}_\delta(x, y) \begin{bmatrix} \tilde{\phi}_1(y) \\ \tilde{\phi}_4(y) \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \tilde{\phi}_1(x) \\ \frac{1}{2} \tilde{\phi}_4(x) \end{bmatrix} \quad (3.8a)$$

and

$$\int_0^1 [dy] \bar{V}_\Delta(x, y) \begin{bmatrix} \tilde{\phi}_1(y) \\ \tilde{\phi}_4(y) \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \tilde{\phi}_1(x) \\ -\frac{17}{6} \tilde{\phi}_4(x) \end{bmatrix}, \quad (3.8b)$$

which gives the anomalous dimensions as ($n_f=3$ is taken)

$$\begin{aligned} \gamma_1 &= \left(2 \times \frac{2}{3} C_B + \frac{3}{2} C_F \right) / \beta = \frac{26}{81} \\ \gamma_4 &= \left(2 \times \frac{7}{3} C_B + \frac{3}{2} C_F \right) / \beta = \frac{46}{81} \end{aligned}$$

and the orbital evolution of the baryon singlet is given by

$$\phi(x, Q^2) = \phi_{as}(x) \sum_{i=0}^5 N_i \left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \right]^{-\gamma_i} a_i A_i(x)$$

where $\phi_{as}(x) = 120 x_1 x_2 x_3$ and $A_i(x)$ are the Appel polynomials. Here the normalizations N_i , a_i are given in Table II and we found $a_0 = a_2 = a_3 = a_5 = 0$. We also applied the same method to the baryon octet and decouplet and explicitly verified the obtained anomalous dimensions of baryon octet and decouplet are the same as those of nucleon and delta obtained in Ref. [3]. For specific calculations, one does not always have to use the full antisymmetric representation since we can use an effective representation in which the helicity configuration is fixed. As an example, we can choose an $\uparrow\downarrow\uparrow$ term for $S_Z = \frac{1}{2}$ state as an effective representation [3]. We summarized the anomalous dimensions and the eigenfunctions expanded in the Appel polynomials for all baryon multiplets of SU(3) flavor symmetry in Table II.

4 Discussions and Future Directions

In this paper, we applied the general method of solving the QCD evolution equation for the baryon systems to the baryon singlet state in SU(3) flavor symmetry. The anomalous dimensions of the baryon singlet obtained in this paper represents those of $\left\{ \frac{N(N-1)(N-2)}{6} \right\}$ multiplet in SU(N) flavor symmetry because the raising and lowering operators of SU(N) flavor symmetry commute with the QCD evolution kernel given by Eqs. (3.4) and (3.5). We found that for this flavor antisymmetric multiplet the quark distribution can only be expanded in terms of $A_1(x)$ and $A_4(x)$ when the power n is considered up to 2. All the other coefficients must be zero from the symmetry. The corresponding anomalous dimensions are given by $\gamma_1 = \frac{26}{81}$ and $\gamma_4 = \frac{46}{81}$ in $n_f=3$ case. Thus the asymptotic high Q^2 behavior of the flavor antisymmetric baryon (e.g. Λ) form factor is given by

$$F(Q^2) \sim \frac{\alpha_s(Q^2)}{(Q^2)^2} \left(\ln \frac{Q^2}{\Lambda^2} \right)^{-\frac{52}{81}}. \quad (4.1)$$

So far, the model quark distribution amplitude at a given Q^2 (e.g. $Q^2 = Q_0^2 \sim 1 \text{ GeV}^2$) for the baryon singlet is not yet given while the baryon octet [6] and decouplet [7] distribution amplitudes are obtained by QCD sum rule techniques. However, any model quark distribution amplitude for the baryon singlet must have zero coefficient for A_0, A_2, A_3 and A_5 terms. At $Q^2 = Q_0^2$, the flavor symmetry breaking effects must be considered and these effects have been studied using QCD sum rule [8]. Since we are restricted to asymptotic high Q^2 , we neglected the higher twist such as quark mass terms in the QCD evolution kernel. If one includes such terms to break the flavor symmetry, the results we obtained should be corrected and each baryon states even in the same multiplet could have different anomalous dimensions. Such flavor symmetry breaking effects including the effects $\langle \bar{s}s \rangle \neq \langle \bar{u}u \rangle = \langle \bar{d}d \rangle$ both in the evolution kernel and in the baryon singlet quark distribution amplitude at $Q^2 = Q_0^2$ are under investigation.

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TABLE I. Completely antisymmetric three-quark representation for (A) $P, N, \Sigma^+, \Sigma^-, \Xi^0$ and Ξ^- systems; (B) Δ, Σ^*, Ξ^* and Ω^- systems; (C) Σ^0 system; (D) Λ_8 system and (E) Λ_1 system in terms of Appel polynomials $A_i(x)(i \leq 5)$ where $A_0(x_1, x_2) = 1, A_1(x_1, x_2) = \frac{21}{2}(x_1 - x_2), A_2(x_1, x_2) = \frac{7}{2}\{2 - 3(x_1 + x_2)\}, A_3(x_1, x_2) = \frac{63}{10}\{2 - 7(x_1 + x_2) + 8(x_1^2 + x_2^2) + 4x_1 x_2\}, A_4(x_1, x_2) = \frac{567}{2}\{x_1 - x_2 - \frac{1}{3}(x_1^2 - x_2^2)\}, A_5(x_1, x_2) = \frac{81}{5}\{2 - 7(x_1 + x_2) + \frac{13}{3}(x_1^2 + x_2^2) + 14x_1 x_2\}.$

(A) $P(a = u, b = d); N(a = d, b = u); \Sigma^+(a = u, b = s); \Sigma^-(a = d, b = s); \Xi^0(a = s, b = u)$ and $\Xi^-(a = s, b = d)$ systems;

Index-Power Symmetry	Representation of the Three-Quark System
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1. $(S, S_Z) = (3/2, 3/2)$

	$\frac{2}{21\sqrt{2}} (\uparrow\uparrow\uparrow) [baa A_2(x_2, x_3) + aba A_2(x_1, x_3) + aab A_2(x_1, x_2)]$
	$\frac{1}{54\sqrt{2}} (\uparrow\uparrow\uparrow) [baa \{-\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3)\} + aba \{-\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3)\} + aab \{-\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2)\}]$

2. $(S, S_Z) = (3/2, 1/2)$

	$\frac{2}{21\sqrt{6}} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] [baa A_2(x_2, x_3) + aba A_2(x_1, x_3) + aab A_2(x_1, x_2)]$
	$\frac{1}{54\sqrt{6}} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] [baa \{-\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3)\} + aba \{-\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3)\} + aab \{-\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2)\}]$

3. $(S, S_Z) = (1/2, 1/2)$

	$\frac{1}{3\sqrt{2}} [(\downarrow\uparrow\uparrow)(2baa - aba - aab) + (\uparrow\downarrow\uparrow)(-baa + 2aba - aab) + (\uparrow\uparrow\downarrow)(-baa - aba + 2aab)] \times A_0(x_1, x_2)$
	$-\frac{2}{21\sqrt{6}} [(\downarrow\uparrow\uparrow)\{baa A_2(x_2, x_3) + aba A_2(x_1, x_2) + aab A_2(x_1, x_3)\} + (\uparrow\downarrow\uparrow)\{baa A_2(x_1, x_2) + aba A_2(x_1, x_3) + aab A_2(x_2, x_3)\} + (\uparrow\uparrow\downarrow)\{baa A_2(x_1, x_3) + aba A_2(x_2, x_3) + aab A_2(x_1, x_2)\}]$

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline \end{array} - \frac{1}{189\sqrt{6}} [((\downarrow\uparrow\uparrow)(2baa - aba - aab) + (\uparrow\downarrow\uparrow)(-baa + 2aba - aab) + (\uparrow\uparrow\downarrow)(-baa - aba + 2aab)) [A_3(x_1, x_3) + \frac{1}{6}A_5(x_1, x_3)]]$$

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 \\ \hline \end{array} - \frac{1}{54\sqrt{6}} ((\downarrow\uparrow\uparrow) [baa \{-\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3)\} + aba \{-\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2)\} + aab \{-\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3)\}] + (\uparrow\downarrow\uparrow) [baa \{-\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2)\} + aba \{-\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3)\} + aab \{-\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3)\}] + (\uparrow\uparrow\downarrow) [baa \{-\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3)\} + aba \{-\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3)\} + aab \{-\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2)\}])$$

(B) $\Delta; \Sigma^*; \Xi^*$ and Ω^- systems. The flavor factor uuu for Δ^{++} ; ddd for Δ^- ; for Ω^- ; $\frac{1}{\sqrt{3}}(uud + udu + duu)$ for Δ^+ ; $\frac{1}{\sqrt{3}}(ddu + dud + udd)$ for Δ^0 ; $\frac{1}{\sqrt{3}}(ssu + sus + uss)$ for Ξ^{*+} ; $\frac{1}{\sqrt{3}}(ssd + sds + dss)$ for Ξ^{*-} ; $\frac{1}{\sqrt{3}}(uus + usu + suu)$ for Σ^{*+} ; $\frac{1}{\sqrt{3}}(dds + dsd + sdd)$ for Σ^{*-} ; $\frac{1}{\sqrt{6}}(uds + dus + sud + usd + dsu + sdu)$ for Σ^{*0} should be multiplied to all representations.

Index-Power Symmetry

Representation of the Three-Quark System

1. $(S, S_Z) = (3/2, 3/2)$

$$\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array} (\uparrow\uparrow\uparrow) \times A_0(x_1, x_3)$$

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline \end{array} - \frac{1}{63\sqrt{3}} (\uparrow\uparrow\uparrow) [A_3(x_1, x_3) + \frac{1}{6}A_5(x_1, x_3)]$$

2. $(S, S_Z) = (3/2, 1/2)$

$$\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array} \frac{1}{3}[(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \times A_0(x_1, x_3)$$

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline \end{array} - \frac{1}{189} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] [A_3(x_1, x_3) + \frac{1}{6}A_5(x_1, x_3)]$$

3. $(S, S_Z) = (1/2, 1/2)$

$$\begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 \\ \hline \end{array} \frac{2}{21\sqrt{2}} [(\downarrow\uparrow\uparrow) A_2(x_2, x_3) + (\uparrow\downarrow\uparrow) A_2(x_1, x_1)]$$

$$+ (\uparrow\uparrow\downarrow) A_2(x_1, x_2)]$$

0	1
1	

$$\frac{1}{54\sqrt{2}} \left[(\downarrow\uparrow\uparrow) \left\{ -\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3) \right\} \right. \\ \left. + (\uparrow\downarrow\uparrow) \left\{ -\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3) \right\} \right. \\ \left. + (\uparrow\uparrow\downarrow) \left\{ -\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2) \right\} \right]$$

(C) Σ^0 system.

Index-Power
Symmetry

Representation of the Three-Quark System

1. $(S, S_Z) = (3/2, 3/2)$

0	0
1	

$$\frac{1}{21} (\uparrow\uparrow\uparrow) [(dsu + usd) A_2(x_2, x_3) \\ + (sud + sdu) A_2(x_1, x_3) \\ + (dus + uds) A_2(x_1, x_2)]$$

0	1
1	

$$\frac{1}{108} (\uparrow\uparrow\uparrow) \left[(dsu + usd) \left\{ -\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3) \right\} \right. \\ \left. + (sud + sdu) \left\{ -\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3) \right\} \right. \\ \left. + (dus + uds) \left\{ -\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2) \right\} \right]$$

2. $(S, S_Z) = (3/2, 1/2)$

0	0
1	

$$\frac{1}{21\sqrt{3}} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] [(dsu + usd) A_2(x_2, x_3) \\ + (sud + sdu) A_2(x_1, x_3) \\ + (dus + uds) A_2(x_1, x_2)]$$

0	1
1	

$$\frac{1}{108\sqrt{3}} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \left[(dsu + usd) \left\{ -\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3) \right\} \right. \\ \left. + (sud + sdu) \left\{ -\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3) \right\} \right. \\ \left. + (dus + uds) \left\{ -\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2) \right\} \right]$$

3. $(S, S_Z) = (1/2, 1/2)$

0	0	0
1		

$$\frac{1}{6} [(\downarrow\uparrow\uparrow) (2(dsu + usd) - (sud + sdu) - (dus + uds)) \\ + (\uparrow\downarrow\uparrow) (- (dsu + usd) + 2(sud + sdu) - (dus + uds)) \\ + (\uparrow\uparrow\downarrow) (- (dsu + usd) - (sud + sdu) + 2(dus + uds))] \times A_{10}(x_1, r_1)$$

0	0
1	

$$-\frac{1}{21\sqrt{3}} [(\downarrow\uparrow\uparrow) \{ (dsu + usd) A_2(x_2, x_3) \\ + (sud + sdu) A_2(x_1, x_2) \}$$

$$\begin{aligned}
& + (dus + uds) A_2(x_1, x_3) \} \\
& + (\uparrow\downarrow\uparrow) \{ (dsu + usd) A_2(x_1, x_2) \\
& \quad + (sud + sdu) A_2(x_1, x_3) \\
& \quad + (dus + uds) A_2(x_2, x_3) \} \\
& + (\uparrow\uparrow\downarrow) \{ (dsu + usd) A_2(x_1, x_3) \\
& \quad + (sud + sdu) A_2(x_2, x_3) \\
& \quad + (dus + uds) A_2(x_1, x_2) \}
\end{aligned}$$

0	1	1
---	---	---

$$\begin{aligned}
& - \frac{1}{378\sqrt{3}} [(\downarrow\uparrow\uparrow)(2(dsu + usd) - (sud + sdu) - (dus + uds)) \\
& \quad + (\uparrow\downarrow\uparrow)(-(dsu + usd) + 2(sud + sdu) - (dus + uds)) \\
& \quad + (\uparrow\uparrow\downarrow)(-(dsu + usd) - (sud + sdu) + 2(dus + uds))] \\
& \times [A_3(x_1, x_3) + \frac{1}{6}A_5(x_1, x_3)]
\end{aligned}$$

0	1
1	

$$\begin{aligned}
& - \frac{1}{108\sqrt{3}} ((\downarrow\uparrow\uparrow) \left[(dsu + usd) \{ -\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3) \} \right. \\
& \quad + (sud + sdu) \{ -\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2) \} \\
& \quad \left. + (dus + uds) \{ -\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3) \} \right] \\
& + (\uparrow\downarrow\uparrow) \left[(dsu + usd) \{ -\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2) \} \right. \\
& \quad + (sud + sdu) \{ -\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3) \} \\
& \quad \left. + (dus + uds) \{ -\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3) \} \right] \\
& + (\uparrow\uparrow\downarrow) \left[(dsu + usd) \{ -\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3) \} \right. \\
& \quad + (sud + sdu) \{ -\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3) \} \\
& \quad \left. + (dus + uds) \{ -\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2) \} \right])
\end{aligned}$$

(D) Λ_8 system.

Index-Power
Symmetry

Representation of the Three-Quark System

1. $(S, S_Z) = (3/2, 3/2)$

0	0
1	

$$\begin{aligned}
& \frac{1}{21\sqrt{3}} (\uparrow\uparrow\uparrow) [(sdu - sud) A_1(x_1, x_3) \\
& \quad - (usd - dsu) A_1(x_2, x_3) \\
& \quad - (dus - uds) A_1(x_1, x_2)]
\end{aligned}$$

0	1
1	

$$\begin{aligned}
& \frac{1}{756\sqrt{3}} (\uparrow\uparrow\uparrow) [-(sdu - sud) A_4(x_1, x_3) \\
& \quad + (usd - dsu) A_4(x_2, x_3) \\
& \quad + (dus - uds) A_4(x_1, x_2)]
\end{aligned}$$

2. $(S, S_Z) = (3/2, 1/2)$

0	0
1	

$$\frac{1}{63} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \quad [(sdu - sud) A_1(x_1, x_3) \\ - (usd - dsu) A_1(x_2, x_3) \\ - (dus - uds) A_1(x_1, x_2)]$$

0	1
1	

$$\frac{1}{2268} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \quad [-(sdu - sud) A_4(x_1, x_3) \\ + (usd - dsu) A_4(x_2, x_3) \\ + (dus - uds) A_4(x_1, x_2)]$$

3. $(S, S_Z) = (1/2, 1/2)$

0	0	0
1		

$$\frac{1}{2\sqrt{3}} [(\downarrow\uparrow\uparrow) \{ (sdu - sud) - (dus - uds) \} \\ + (\uparrow\downarrow\uparrow) \{ -(usd - dsu) + (dus - uds) \} \\ + (\uparrow\uparrow\downarrow) \{ -(sdu - sud) + (usd - dsu) \}] \times A_0(x_1, x_2)$$

0	0
1	

$$\frac{1}{63} [(\downarrow\uparrow\uparrow) \{ -(sdu - sud) A_1(x_1, x_2) \\ - (usd - dsu) A_1(x_2, x_3) \\ + (dus - uds) A_1(x_1, x_3) \} \\ + (\uparrow\downarrow\uparrow) \{ -(sdu - sud) A_1(x_1, x_3) \\ - (usd - dsu) A_1(x_1, x_2) \\ - (dus - uds) A_1(x_2, x_3) \} \\ + (\uparrow\uparrow\downarrow) \{ -(sdu - sud) A_1(x_2, x_3) \\ + (usd - dsu) A_1(x_1, x_3) \\ - (dus - uds) A_1(x_1, x_2) \}]$$

0	1	1
1		

$$-\frac{1}{378} [(\downarrow\uparrow\uparrow) \{ (sdu - sud) - (dus - uds) \} \\ + (\uparrow\downarrow\uparrow) \{ -(usd - dsu) + (dus - uds) \} \\ + (\uparrow\uparrow\downarrow) \{ -(sdu - sud) + (usd - dsu) \}] \\ \times [A_3(x_1, x_3) + \frac{1}{6} A_5(x_1, x_3)]$$

0	1
1	

$$\frac{1}{2268} [(\downarrow\uparrow\uparrow) [(sdu - sud) A_4(x_1, x_2) \\ + (usd - dsu) A_4(x_2, x_3) \\ - (dus - uds) A_4(x_1, x_3)] \\ + (\uparrow\downarrow\uparrow) [-(sdu - sud) A_4(x_1, x_3) \\ + (usd - dsu) A_4(x_1, x_2) \\ + (dus - uds) A_4(x_2, x_3)] \\ + (\uparrow\uparrow\downarrow) [(sdu - sud) A_4(x_2, x_3) \\ - (usd - dsu) A_4(x_1, x_3) \\ + (dus - uds) A_4(x_1, x_2)]]$$

(E) Λ_1 system. The flavor factor $-\frac{1}{\sqrt{6}}[(usd - dsu) + (sdu - sud) + (dus - uds)]$ should be multiplied to all representations.

Index-Power
Symmetry

Representation of the Three-Quark System

1. $(S, S_Z) = (1/2, 1/2)$

0	0
1	

$$\frac{2}{21\sqrt{6}} [-(\downarrow\uparrow\uparrow) A_1(x_2, x_3) + (\uparrow\downarrow\uparrow) A_1(x_1, x_3) - A_1(x_1, x_2)]$$

0	1
1	

$$\frac{1}{378\sqrt{6}} [(\downarrow\uparrow\uparrow) A_4(x_2, x_3) - (\uparrow\downarrow\uparrow) A_4(x_1, x_3) + (\uparrow\uparrow\downarrow) A_4(x_1, x_2)]$$

TABLE II. Eigenvalues and eigensolutions for (A) $P, N, \Sigma^+, \Sigma^-, \Xi^0$ and Ξ^- systems; (B) Δ, Σ^*, Ξ^* and Ω^- systems; (C) Σ^0 system; (D) Λ_8 system and (E) Λ_1 system in terms of Appel polynomials. The anomalous dimensions are related to the values of b by $\gamma = (2bC_B + 3C_F/2)/\beta$.

(A) $P(a = u, b = d); N(a = d, b = u); \Sigma^+(a = u, b = s); \Sigma^-(a = d, b = s); \Xi^0(a = s, b = u)$ and $\Xi^-(a = s, b = d)$ systems.

Spin Configuration	b	Spin \times Orbital	Normalization	$\phi(y)$ (Effective Representation)
$\uparrow\uparrow\uparrow$	$\frac{3}{2}$	$\begin{array}{ c c c } \hline \uparrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array}$	$\sqrt{21} \times \sqrt{5!}$	$\frac{2}{21\sqrt{2}} [baa A_2(x_2, x_3) + aba A_2(x_1, x_3) + aab A_2(x_1, x_2)]$
$\uparrow\uparrow\uparrow$	$\frac{17}{6}$	$\begin{array}{ c c c } \hline \uparrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$	$6\sqrt{28} \times \sqrt{5!}$	$\frac{1}{54\sqrt{2}} [baa \{-\frac{1}{7}A_3(x_2, x_3) + \frac{1}{3}A_5(x_2, x_3)\} + aba \{-\frac{1}{7}A_3(x_1, x_2) + \frac{1}{3}A_5(x_1, x_2)\} + aab \{-\frac{1}{7}A_3(x_1, x_3) + \frac{1}{3}A_5(x_1, x_3)\}]$
$\uparrow\downarrow\uparrow$	-1	$\begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c c } \hline 0 & 0 & 0 \\ \hline \end{array}$	$\sqrt{5!}$	$\frac{1}{\sqrt{6}} (-baa + 2aba - aab) \times A_0(x_1, x_3)$
$\uparrow\downarrow\uparrow$	$\frac{2}{3}$	$\frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array}$ + $\frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array}$	$\sqrt{21} \times \sqrt{5!}$	$\frac{1}{21} (baa - aab) A_1(x_1, x_3)$
$\uparrow\downarrow\uparrow$	1	$-\frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array}$ + $\frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array}$	$\sqrt{21} \times \sqrt{5!}$	$\frac{1}{21} (baa - 2aba + aab) \times A_2(x_1, x_3)$

Spin Configuration	b	Spin x Orbital	Normalization	$\phi(y)$ (Effective Representation)
$\uparrow\downarrow\uparrow$	$\frac{5}{3}$	$\frac{1}{\sqrt{30}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$ $-\frac{\sqrt{28}}{\sqrt{30}} \begin{array}{ c c c } \hline \uparrow & \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c c } \hline 0 & 1 & 1 \\ \hline \end{array}$ $-\frac{1}{\sqrt{30}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$	$6\sqrt{28} \times \sqrt{5!}$	$\frac{5}{252\sqrt{15}} (baa - 2aba + aab) \times A_3(x_1, x_3)$
$\uparrow\downarrow\uparrow$	$\frac{7}{3}$	$\frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$ $+\frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$	$6\sqrt{28} \times \sqrt{5!}$	$-\frac{1}{756} (baa - aab) \times A_4(x_1, x_3)$
$\uparrow\downarrow\uparrow$	$\frac{5}{2}$	$-\frac{1}{\sqrt{15}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$ $-\frac{1}{\sqrt{15}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 & 1 \\ \hline \end{array}$ $+\frac{1}{\sqrt{15}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$	$\frac{42}{\sqrt{15}} \sqrt{5!}$	$\frac{5}{378\sqrt{2}} (baa - 2aba + aab) \times A_5(x_1, x_3)$

(B) Δ ; Σ^* ; Ξ^* and Ω^- systems.

The flavor factor uuu for Δ^{++} ; ddd for Δ^- ; sss for Ω^- ;

$\frac{1}{\sqrt{3}}(uud + udu + duu)$ for Δ^+ ; $\frac{1}{\sqrt{3}}(ddu + dud + udd)$ for Δ^0 ; $\frac{1}{\sqrt{3}}(ssu + sus + uss)$ for Ξ^{*+} ;
 $\frac{1}{\sqrt{3}}(ssd + sds + dss)$ for Ξ^{*-} ; $\frac{1}{\sqrt{3}}(uus + usu + suu)$ for Σ^{*+} ; $\frac{1}{\sqrt{3}}(dds + dsd + sdd)$ for Σ^{*-} ;
 $\frac{1}{\sqrt{6}}(uds + dus + sud + usd + dsu + sd)$ for Σ^{*0} should be multiplied to all representations.

Spin Configura-	b	Spin \times Orbital ration	Normalization	$\phi(y)$ (Effective Representation)
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$$\uparrow\uparrow\uparrow \quad 0 \quad \boxed{\uparrow\uparrow\uparrow} \times \boxed{000} \quad \sqrt{5!} \quad A_0(\mathbf{z}_1, \mathbf{z}_3)$$

$$\uparrow\downarrow\uparrow \quad \frac{7}{3} \quad \boxed{\uparrow\uparrow\uparrow} \times \boxed{011} \quad 42\sqrt{5!} \quad -\frac{1}{63\sqrt{3}} [A_3(\mathbf{z}_1, \mathbf{z}_3) + \frac{1}{6}A_5(\mathbf{z}_1, \mathbf{z}_3)]$$

$$\uparrow\downarrow\uparrow \quad -1 \quad \boxed{\uparrow\downarrow\uparrow} \times \boxed{000} \quad \sqrt{5!} \quad A_0(\mathbf{z}_1, \mathbf{z}_3)$$

$$\uparrow\downarrow\uparrow \quad 1 \quad \boxed{\substack{\uparrow\uparrow \\ \downarrow}} \times \boxed{00} \quad \sqrt{21} \times \sqrt{5!} \quad \frac{2}{7\sqrt{6}} A_2(\mathbf{z}_1, \mathbf{z}_3)$$

$$\begin{aligned} \uparrow\downarrow\uparrow \quad \frac{5}{3} \quad & \sqrt{\frac{14}{15}} \boxed{\uparrow\downarrow\uparrow} \times \boxed{011} \\ & + \sqrt{\frac{1}{15}} \boxed{\substack{\uparrow\uparrow \\ \downarrow}} \times \boxed{01} \end{aligned} \quad \frac{6\sqrt{14}\times\sqrt{5!}}{\sqrt{15}} \quad \frac{5\sqrt{3}}{126} A_3(\mathbf{z}_1, \mathbf{z}_3)$$

$$\begin{aligned} \uparrow\downarrow\uparrow \quad \frac{5}{2} \quad & -\sqrt{\frac{1}{15}} \boxed{\uparrow\downarrow\uparrow} \times \boxed{011} \\ & + \sqrt{\frac{14}{15}} \boxed{\substack{\uparrow\uparrow \\ \downarrow}} \times \boxed{01} \end{aligned} \quad \frac{42\times\sqrt{5!}}{\sqrt{15}} \quad -\frac{5}{126\sqrt{3}} A_5(\mathbf{z}_1, \mathbf{z}_3)$$

(C) Σ^0 system

Spin Configuration	b	Spin x Orbital	Normalization	$\phi(y)$ (Effective Representation)
$\uparrow\uparrow\uparrow$	$\frac{3}{2}$	$\begin{array}{ c c c } \hline \uparrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array}$	$\sqrt{21} \times \sqrt{5!}$	$\frac{1}{21} [(dsu + usd) A_2(x_2, x_3) + (sud + sdu) A_2(x_1, x_3) + (uds + dus) A_2(x_1, x_2)]$
$\uparrow\uparrow\uparrow$	$\frac{17}{6}$	$\begin{array}{ c c c } \hline \uparrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$	$6\sqrt{28} \times \sqrt{5!}$	$\frac{1}{108} [(dsu + usd) \{-\frac{1}{7} A_3(x_2, x_3)\} + \frac{1}{3} A_5(x_2, x_3) + (sud + sdu) \{1 \leftrightarrow 3\} + (uds + dus) \{1 \leftrightarrow 2\}]$
$\uparrow\downarrow\uparrow$	-1	$\begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c c } \hline 0 & 0 & 0 \\ \hline 1 \\ \hline \end{array}$	$\sqrt{5!}$	$\frac{1}{2\sqrt{3}} \{-(dsu + usd) + 2(sud + sdu) - (uds + dus)\} \times A_0(x_1, x_3)$
$\uparrow\downarrow\uparrow$	$\frac{2}{3}$	$\begin{aligned} & \frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \downarrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array} \\ & -\frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array} \end{aligned}$	$\sqrt{21} \times \sqrt{5!}$	$\frac{1}{21\sqrt{2}} \{(dsu + usd) - (uds + dus)\} A_1(x_1, x_3)$
$\uparrow\downarrow\uparrow$	1	$\begin{aligned} & -\frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \downarrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array} \\ & + \frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array} \end{aligned}$	$\sqrt{21} \times \sqrt{5!}$	$\frac{1}{7\sqrt{3}} \{(dsu + usd) - 2(sud + sdu) + (uds + dus)\} \times A_2(x_1, x_3)$
$\uparrow\downarrow\uparrow$	$\frac{5}{3}$	$\begin{aligned} & \frac{1}{\sqrt{30}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \downarrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array} \\ & -\frac{\sqrt{28}}{\sqrt{30}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c c } \hline 0 & 1 & 1 \\ \hline 1 \\ \hline \end{array} \\ & -\frac{\sqrt{5}}{\sqrt{30}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array} \end{aligned}$	$6\sqrt{28} \times \sqrt{5!}$	$\frac{5}{252\sqrt{30}} \{(dsu + usd) - 2(sud + sdu) + (uds + dus)\} \times A_3(x_1, x_3)$

Spin Configuration	b	Spin x Orbital	Normalization	$\phi(y)$ (Effective Representation)
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$$\begin{array}{c} \uparrow\downarrow\uparrow \\ \text{Spin Configuration} \\ \uparrow\downarrow\uparrow \end{array} \quad b = \frac{7}{3} \quad \begin{array}{c} \frac{1}{\sqrt{2}} \begin{array}{|c|c|c|} \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \\ + \frac{1}{\sqrt{2}} \begin{array}{|c|c|} \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \end{array} \quad 6\sqrt{28} \times \sqrt{5!} \quad -\frac{1}{756\sqrt{2}} \{ (dsu + usd) - (uds + dus) \} \\ \times A_4(x_1, x_3)$$

$$\begin{array}{c} \uparrow\downarrow\uparrow \\ \text{Spin Configuration} \\ \uparrow\downarrow\uparrow \end{array} \quad b = \frac{5}{2} \quad \begin{array}{c} -\frac{1}{\sqrt{15}} \begin{array}{|c|c|c|} \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \\ -\frac{1}{\sqrt{15}} \begin{array}{|c|c|} \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline \end{array} \\ + \frac{1}{\sqrt{15}} \begin{array}{|c|c|} \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \end{array} \quad \frac{42}{\sqrt{15}} \sqrt{5!} \quad \frac{5}{756} \{ (dsu + usd) - 2(sud + sdu) \\ + (uds + dus) \} \times A_5(x_1, x_3)$$

(D) Λ_8 system.

Spin Configuration	b	Spin x Orbital	Normalization	$\phi(y)$ (Effective Representation)
$\uparrow\uparrow\uparrow$	$\frac{3}{2}$	$\begin{array}{ c c c } \hline \uparrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array}$	$\sqrt{21} \times \sqrt{5!}$	$\frac{1}{21\sqrt{3}} [(sdu - sud) A_1(z_1, z_3) - (usd - dsu) A_1(z_2, z_3) - (dus - uds) A_1(z_1, z_2)]$
$\uparrow\uparrow\uparrow$	$\frac{17}{6}$	$\begin{array}{ c c c } \hline \uparrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$	$6\sqrt{28} \times \sqrt{5!}$	$\frac{1}{756\sqrt{3}} [- (sdu - sud) A_4(z_1, z_3) + (usd - dsu) A_4(z_2, z_3) + (dus - uds) A_4(z_1, z_2)]$
$\uparrow\downarrow\uparrow$	-1	$\begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c c } \hline 0 & 0 & 0 \\ \hline \end{array}$	$\sqrt{5!}$	$\frac{1}{2} \{ - (usd - dsu) + (dus - uds) \} \times A_0(z_1, z_3)$
$\uparrow\downarrow\uparrow$	$\frac{2}{3}$	$\begin{array}{l} \frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array} \\ + \frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array} \end{array}$	$\sqrt{21} \times \sqrt{5!}$	$\frac{1}{21\sqrt{6}} \{ 2(sdu - sud) - (usd - dsu) - (dus - uds) \} \times A_1(z_1, z_3)$
$\uparrow\downarrow\uparrow$	1	$\begin{array}{l} -\frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array} \\ + \frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 \\ \hline \end{array} \end{array}$	$\sqrt{21} \times \sqrt{5!}$	$\frac{1}{7\sqrt{6}} \{ (usd - dsu) - (dus - uds) \} \times A_2(z_1, z_3)$
$\uparrow\downarrow\uparrow$	$\frac{5}{3}$	$\begin{array}{l} \frac{1}{\sqrt{30}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array} \\ -\frac{\sqrt{24}}{\sqrt{30}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c c } \hline 0 & 1 & 1 \\ \hline \end{array} \\ -\frac{1}{\sqrt{30}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array} \end{array}$	$6\sqrt{28} \times \sqrt{5!}$	$-\frac{5}{84\sqrt{3}} \{ (usd - dsu) - (dus - uds) \} \times A_3(z_1, z_3)$

Spin
Configu- b Spin × Orbital Normalization $\phi(y)$ (Effective Representation)
ration

$$\begin{array}{ll} \text{Spin} & \frac{1}{\sqrt{2}} \begin{array}{|c|c|c|} \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \\ \text{Configu-} & + \frac{1}{\sqrt{2}} \begin{array}{|c|c|c|} \hline \uparrow & \uparrow & \uparrow \\ \hline \downarrow & & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \\ \text{ration} & \end{array} \quad 6\sqrt{28} \times \sqrt{5!} \quad \frac{1}{756\sqrt{3}} \{ -2(sdu - sud) + (usd - dsu) \\ & + (dus - uds) \} \times A_4(x_1, x_3)$$

$$\begin{array}{ll} \text{Spin} & -\frac{1}{\sqrt{15}} \begin{array}{|c|c|c|} \hline \uparrow & \downarrow & \uparrow \\ \hline \downarrow & & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \\ \text{Configu-} & -\frac{1}{\sqrt{15}} \begin{array}{|c|c|c|} \hline \uparrow & \uparrow & \uparrow \\ \hline \downarrow & & \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline & & \\ \hline \end{array} \\ \text{ration} & + \frac{1}{\sqrt{15}} \begin{array}{|c|c|c|} \hline \uparrow & \uparrow & \uparrow \\ \hline \downarrow & & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \\ & \end{array} \quad \frac{42}{\sqrt{15}} \sqrt{5!} \quad \frac{1}{28\sqrt{3}} \{ (usd - dsu) - (dus - uds) \} \\ & \times [4A_0(x_1, x_3) + A_2(x_1, x_3) \\ & - \frac{1}{9}A_3(x_1, x_3) + \frac{2}{27}A_5(x_1, x_3)]$$

(E) Λ_1 system. The flavor factor $-\frac{1}{\sqrt{6}}[(usd - dsu) + (sdu - sud) + (dus - uds)]$
should be multiplied to all representations.

Spin
Configu- b Spin × Orbital Normalization $\phi(y)$ (Effective Representation)
ration

$$\begin{array}{ll} \text{Spin} & \begin{array}{|c|c|c|} \hline \uparrow & \downarrow & \uparrow \\ \hline \uparrow & & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} \\ \text{Configu-} & \end{array} \quad \sqrt{21} \times \sqrt{5!} \quad -\frac{1}{21\sqrt{3}} \{ (sdu - sud) + (usd - dsu) \\ \text{ration} & + (dus - uds) \} \times A_1(x_1, x_3)$$

$$\begin{array}{ll} \text{Spin} & \begin{array}{|c|c|c|} \hline \uparrow & \downarrow & \uparrow \\ \hline \uparrow & & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \\ \text{Configu-} & \end{array} \quad 6\sqrt{28} \times \sqrt{5!} \quad \frac{1}{756\sqrt{3}} \{ (sdu - sud) + (usd - dsu) \\ \text{ration} & + (dus - uds) \} \times A_1(x_1, x_3)$$